

Uniformly resolvable decompositions of K_v into paths on two, three and four vertices

Giovanni Lo Faro *

Dipartimento di Matematica e Informatica

Università di Messina

Messina

Italia

`lofaro@unime.it`

Salvatore Milici †

Dipartimento di Matematica e Informatica

Università di Catania

Catania

Italia

`milici@dmi.unict.it`

Antoinette Tripodi ‡

Dipartimento di Matematica e Informatica

Università di Messina

Messina

Italia

`atripodi@unime.it`

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Abstract

In this paper we consider uniformly resolvable decompositions of the complete graph K_v into subgraphs such that each resolution class contains only blocks isomorphic to the same graph. We completely determine the spectrum for the case in which all the resolution classes consist of either P_2 , P_3 and P_4 .

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1 Introduction and definitions

Given a collection \mathcal{H} of graphs, an \mathcal{H} -*decomposition* of a graph G is a decomposition of the edge set of G into subgraphs (called *blocks*) isomorphic to some element of \mathcal{H} . Such a decomposition is said to be *resolvable* if it is possible to partition the blocks into classes \mathcal{P}_i (often referred to as *parallel classes*) such that every vertex of G appears in exactly one block of each \mathcal{P}_i . A resolvable \mathcal{H} -decomposition of G is sometimes also referred to as an \mathcal{H} -*factorization* of G , and a class can be called an \mathcal{H} -*factor* of G . The case where $\mathcal{H} = \{K_2\}$ (a single edge) is known as a *1-factorization*; for $G = K_v$ it is well known to exist if and only if v is even. A single class of a 1-factorization, that is a pairing of all vertices, is also known as a *1-factor* or *perfect matching*.

In many cases we wish to place further constraints on the classes. For example, a class is called *uniform* if every block of the class is isomorphic to the same graph from \mathcal{H} . The result of Rees [16] which finds necessary and sufficient conditions for the existence of uniformly resolvable $\{K_2, K_3\}$ -decompositions of K_v is of particular note. Uniformly resolvable decompositions of K_v have also been studied in [4], [5], [6], [7], [8], [9], [11], [12], [13], [15], [18], [19], [20] and [21].

If $\mathcal{H} = \{H_1, H_2, \dots, H_l\}$, let (H_1, H_2, \dots, H_l) -URD($v; r_1, r_2, \dots, r_l$) denote a uniformly resolvable decomposition of K_v into r_i classes containing only copies of the graph H_i , for $i = 1, 2, \dots, l$. In this paper we study the existence of uniformly resolvable decompositions into paths $P_2 = K_2$, P_3 , P_4 for the complete graph K_v . The existence of (uniformly) resolvable decompositions for each $\mathcal{H} \subset \{K_2, P_3, P_4\}$ was studied separately already long ago:

- There exists a resolvable K_2 -decomposition of K_v if and only if $v \equiv 0 \pmod{2}$.

- There exists a resolvable P_3 -decomposition of K_v if and only if $v \equiv 9 \pmod{12}$ [10].
- There exists a resolvable P_4 -decomposition of K_v if and only if $v \equiv 4 \pmod{12}$ [1].
- There exists a (K_2, P_3) -URD($v; r, s$) if and only if $v \equiv 0 \pmod{6}$ and $(r, s) \in \{(v-1-4x, 3x), x=0, 1, \dots, \frac{v-4}{4}\}$ for $v \equiv 0 \pmod{12}$ and $(r, s) \in \{(v-1-4x, 3x), x=0, 1, \dots, \frac{v-2}{4}\}$ for $v \equiv 6 \pmod{12}$ [14, 24].
- There exists a (K_2, P_4) -URD($v; r, s$) if and only if $v \equiv 0 \pmod{4}$ and $(r, s) \in \{(v-1-3x, 2x), x=0, 1, \dots, \frac{v-3}{3}\}$ for $v \equiv 0 \pmod{12}$, $(r, s) \in \{(v-1-3x, 2x), x=0, 1, \dots, \frac{v-1}{3}\}$ for $v \equiv 4 \pmod{12}$ and $(r, s) \in \{(v-1-3x, 2x), x=0, 1, \dots, \frac{v-2}{3}\}$ for $v \equiv 8 \pmod{12}$ [14, 24].
- There exists a (P_3, P_4) -URD($v; r, s$) if and only if $v \equiv 0 \pmod{12}$ and $(r, s) \in \{(6+9x, 2+\frac{2(v-12)}{3}-8x), x=0, 1, \dots, \frac{v-12}{12}\}$ [8].

In what follows, we will denote by $[a_1, a_2, \dots, a_k]$ the path P_k , $k \geq 3$, having vertex set $\{a_1, a_2, \dots, a_k\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{k-1}, a_k\}\}$. Given a graph G , we will denote by $G_{(n)}$ the graph on $V(G) \times Z_n$ with edge set $\{\{x_i, y_j\} : \{x, y\} \in \mathcal{E}(G), i, j \in Z_n\}$.

2 Necessary conditions

In this section we will give necessary conditions for the existence of a (P_2, P_3, P_4) -URD($v; r, s, t$). To begin with, note that if there exists a (P_2, P_3, P_4) -URD($v; r, s, t$) with $s > 0$ and $t > 0$, then $v \equiv 0 \pmod{12}$.

Lemma 2.1. [14, 24] *There exists a (P_2, P_3, P_4) -URD($v; r, 0, t$) if and only if $v \equiv 0 \pmod{4}$ and $(r, 0, t) \in \{(v-1-3x, 0, 2x), x=0, 1, \dots, \frac{v-3}{3}\}$ for $v \equiv 0 \pmod{12}$, $(r, 0, t) \in \{(v-1-3x, 0, 0), x=0, 1, \dots, \frac{v-1}{3}\}$ for $v \equiv 4 \pmod{12}$ and $(r, 0, t) \in \{(v-1-3x, 0, 2x), x=0, 1, \dots, \frac{v-2}{3}\}$ for $v \equiv 8 \pmod{12}$.*

Lemma 2.2. [14, 24] *There exists a (P_2, P_3, P_4) -URD($v; r, s, 0$) if and only if $v \equiv 0 \pmod{6}$ and $(r, s, 0) \in \{(v-1-4x, 3x, 0), x=0, 1, \dots, \frac{v-4}{4}\}$ for $v \equiv 0 \pmod{12}$ and $(r, s, 0) \in \{(v-1-4x, 3x, 0), x=0, 1, \dots, \frac{v-2}{4}\}$ for $v \equiv 6 \pmod{12}$.*

Lemma 2.3. *Let $v \equiv 2, 10 \pmod{12}$. A (P_2, P_3, P_4) -URD($v; r, s, t$) there exists if and only if $s = t = 0$.*

Proof. Suppose there exists a (P_2, P_3, P_4) -URD($v; r, s, t$), with $(s, t) \neq (0, 0)$. By the resolvability v must be divisible by 4, 6 or 12. A contradiction. \square

Given $v \equiv 0 \pmod{12}$, for every $0 \leq u \leq \frac{v-1}{4}$ define $r(v, u)$ according to the following table:

u	$r(v, u)$
$0 \pmod{3}$	$\frac{v-3}{3} - \frac{4u}{3}$
$1 \pmod{3}$	$\frac{v-6}{3} - \frac{4(u-1)}{3}$
$2 \pmod{3}$	$\frac{v-9}{3} - \frac{4(u-2)}{3}$

Table 1: $r(v, u)$

and let

$$D(v) = \bigcup_{i=0}^{\frac{v-4}{4}} D_{i(v)}, \quad (1)$$

where

$$D_i(v) = \{(v-1-4i-3y, 3i, 2y), 0 \leq y \leq r(v, i)\}, \quad (2)$$

for $i = 0, 1, \dots, \frac{v-4}{4}$.

Lemma 2.4. *Let $v \equiv 0 \pmod{12}$. If there exists a (P_2, P_3, P_4) -URD($v; r, s, t$) then $(r, s, t) \in D(v)$.*

Proof. Assume that there exists a (P_2, P_3, P_4) -URD($v; r, s, t$) \mathcal{D} , $s > 0, t > 0$. By the resolvability,

$$\frac{rv}{2} + \frac{2sv}{3} + \frac{3tv}{4} = \frac{v(v-1)}{2}$$

and hence

$$6r + 8s + 9t = 6(v-1). \quad (3)$$

which implies that $s \equiv 0 \pmod{3}$ and $t \equiv 0 \pmod{2}$. Let $s = 3x$ and $t = 2y$; the equation (1) gives $r = v-1-4x-3y$. Since r, s and t cannot be negative, the value of x and y are in the range as given in the definition of $D(v)$. \square

Let now $URD(v; P_2, P_3, P_4) := \{(r, s, t) : \exists (P_2, P_3, P_4)\text{-URD}(v; r, s, t)\}$. In this paper we completely solve the spectrum problem for such systems, i.e., characterize the existence of uniformly resolvable decompositions of K_v into r 1-factors, s classes containing only copies of P_3 and t classes containing only copies of P_4 by proving the following result:

Main Theorem. *For every integer $v \equiv 0 \pmod{12}$, $URD(v; K_2, P_3, P_4) = D(v)$.*

3 Costructions and related structures

In this section we will introduce some useful definitions and results and discuss constructions we will use in proving the main result. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [3] and its online updates. For some results below, we also cite this handbook instead of the original papers.

An incomplete resolvable (K_2, P_3, P_4) -decomposition of K_{v+h} with a hole of size h is a (K_2, P_3, P_4) -decomposition of $K_{v+h} - K_h$ in which there are two types of classes, *partial* classes which cover every point except those in the hole (the set of points of K_h are referred to as the *hole*) and *full* classes which cover every point of K_{v+h} . Specifically a (K_2, P_3, P_4) -IURD($v+h, h; [r_1, s_1, t_1], [\bar{r}_1, \bar{s}_1, \bar{t}_1]$) is a uniformly resolvable (K_2, P_3, P_4) -decomposition of $K_{v+h} - K_h$ with r_1 1-factors, s_1 classes of copies of P_3 and t_1 classes of copies of P_4 , which cover only the points not in the hole, \bar{r}_1 1-factors, \bar{s}_1 classes of copies of P_3 and \bar{t}_1 classes of copies of P_4 , which cover every point of K_{v+h} .

A (resolvable) \mathcal{H} -decomposition of the complete multipartite graph with u parts each of size g is known as a (resolvable) group divisible design \mathcal{H} -(R)GDD of type g^u (the parts of size g are called the *groups* of the design). When $\mathcal{H} = K_n$ we will call it an n -(R)GDD. A (P_2, P_3, P_4) -URGDD (r, s, t) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r 1-factors, s classes containing only copies of P_3 and t classes containing only copies of P_4 . If the blocks of an \mathcal{H} -GDD of type g^u can be partitioned into partial parallel classes, each of which contains all points except those of one group, we refer to the decomposition as a *frame*. When $\mathcal{H} = K_n$ we will call it an n -*frame* and it is easy to deduce that the number of partial parallel classes missing a specified group G is $\frac{|G|}{n-1}$. We quote the following lemma for a later use.

Define $\bar{r}(k, u)$ according to the following table:

u	$\bar{r}(k, u)$
$0 \pmod{3}$	$4k - \frac{4u}{3}$
$1 \pmod{3}$	$4k - 2 - \frac{4(u-1)}{3}$
$2 \pmod{3}$	$4k - 3 - \frac{4(u-2)}{3}$

Table 2: $\bar{r}(v, u)$

and let

$$\bar{D}((12k)^m) = \bigcup_{j=0}^{3k(m-1)} \bar{D}_j((12k)^m), \quad (4)$$

where

$$\bar{D}_j((12k)^m) = \{(12k(m-1) - 4j - 3y, 3j, 2y), 0 \leq y \leq \bar{r}(k(m-1), j)\}, \quad (5)$$

for $j = 0, 1, \dots, 3k(m-1)$.

Lemma 3.1. *If there exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^k , $s > 0, t > 0$, then $(r, s, t) \in \bar{D}(12^k)$.*

Proof. Assume there exists a (P_2, P_3, P_4) -URGDD(r, s, t) \mathcal{D} , $s > 0, t > 0$. By the resolvability of \mathcal{D}

$$6kr + 8ks + 9kt = 72k(k-1)$$

and hence

$$6r + 8s + 9t = 72(k-1). \quad (6)$$

which implies that $(r, s, t) \in \bar{D}(12^k)$. \square

Let (r_1, s_1, t_1) and (r_2, s_2, t_2) be two triples of non-negative integers. Define $(r_1, s_1, t_1) + (r_2, s_2, t_2) = (r_1 + r_2, s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of triples of non-negative integers, then $X + Y$ denotes the set $\{(r_1, s_1, t_1) + (r_2, s_2, t_2) : (r_1, s_1, t_1) \in X, (r_2, s_2, t_2) \in Y\}$. If X is a set of triples of non-negative integers and h is a positive integer, then $h * X$ denotes the set of all triples of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed). To obtain our main result we will use the following lemmas.

Lemma 3.2. *For every $h \geq 1$, $h * \bar{D}(12^2) = \bar{D}((12h)^2)$.*

Proof. We induct on h . The case $h = 1$ is trivially true. Suppose the assertion holds for $h > 1$ and prove it for $h + 1$. We have

$$(h + 1) * \bar{D}(12^2) = h * \bar{D}(12^2) + \bar{D}(12^2).$$

By induction hypothesis

$$h * \bar{D}(12^2) = \bar{D}((12h)^2) = \bar{D}_0((12h)^2) \cup \bar{D}_1((12h)^2) \cup \dots \cup \bar{D}_{3h}((12h)^2).$$

Since

$$\bar{D}(12^2) = \bar{D}_0(12^2) \cup \bar{D}_1(12^2) \cup \bar{D}_2(12^2) \cup \bar{D}_3(12^2),$$

it is easy to check that

$$\begin{aligned} h * \bar{D}(12^2) + \bar{D}(12^2) &= \\ &= \left\{ \bigcup_{j=0}^{3h} [\bar{D}_j((12h)^2) + \bar{D}_0(12^2)] \right\} \cup \left\{ \bigcup_{i=0}^3 [\bar{D}_i(12^2) + \bar{D}_{3h}((12h)^2)] \right\}. \end{aligned}$$

Now, as it is easy to see, for each $j = 0, 1, \dots, 3h$

$$\bar{D}_j((12h)^2) + \bar{D}_0(12^2) = \bar{D}_j((12h + 12)^2)$$

and, for each $i = 0, 1, 2, 3$,

$$\bar{D}_i(12^2) + \bar{D}_{3h}((12h)^2) = \bar{D}_{i+3h}((12h + 12)^2),$$

and so we obtain

$$h * \bar{D}(12^2) + \bar{D}(12^2) = \bigcup_{j=0}^{3+3h} \bar{D}_j((12h + 12)^2) = \bar{D}((12(h + 1))^2). \quad \square$$

Lemma 3.3. *For every $h \geq 1$, $h * \bar{D}(12^2) = \bar{D}(12^{h+1})$.*

Proof. It is easy to check that $\bar{D}((12h)^2) = \bar{D}(12^{h+1})$ and so the assertion holds by Lemma 3.2. \square

Theorem 3.4. *Let v, g, t, k and u be non-negative integers such that $v = gtu$. If there exists*

- (1) *a k -RGDD of type g^u ;*

(2) a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type t^k with $(r_1, s_1, t_1) \in J_1$;

(3) a (P_2, P_3, P_4) -URD($gt; r_2, s_2, t_2$), with $(r_2, s_2, t_2) \in J_2$;

then there exists a (P_2, P_3, P_4) -URD($v; r, s, t$) for each $(r, s, t) \in J_2 + h * J_1$, where $h = \frac{g(u-1)}{2}$ is the number of parallel classes of the k -RGDD of type g^u .

Proof. Let \mathcal{G} be a k -RGDD of type g^u , with u groups G_i , $i = 1, 2, \dots, u$, of size g ; let $R_1, R_2, \dots, R_{\frac{g(u-1)}{k-1}}$ be the parallel classes of this k -RGDD. Expand each point t times and for each block b of a given resolution class of \mathcal{G} place on $b \times \{1, 2, \dots, t\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type t^k with $(r_1, s_1, t_1) \in J_1$. For each $i = 1, 2, \dots, u$, place on $G_i \times \{1, 2, \dots, t\}$ a copy of a (P_2, P_3, P_4) -URD($gt; r_2, s_2, t_2$) with $(r_2, s_2, t_2) \in J_2$. The result is a (P_2, P_3, P_4) -URD($v; r, s, t$) with $(r, s, t) \in \{J_2 + (\frac{g(u-1)}{k-1}) * J_1\}$. \square

Theorem 3.5. Let v, g, t, h and u be non-negative integers such that $v = gtu + h$. If there exists

(1) a 2-frame \mathcal{F} of type g^u ;

(2) a (K_2, P_3, P_4) -URD($h; r_1, s_1, t_1$) with $(r_1, s_1, t_1) \in J_1$;

(3) a (K_2, P_3, P_4) -URGDD(r_2, s_2, t_2) of type t^2 with $(r_2, s_2, t_2) \in J_2$;

(4) a (K_2, P_3, P_4) -IURD($gt + h, h; [r_1, s_1, t_1], [r_3, s_3, t_3]$) with $(r_1, s_1, t_1) \in J_1$ and $(r_3, s_3, t_3) \in J_3 = g * J_2$;

then exists a (K_2, P_3, P_4) -URD($v + h; r, s, t$) for each $(r, s, t) \in J_1 + u * J_3$.

Proof. Let \mathcal{F} be a 2-frame of type g^u with groups G_i , $i = 1, 2, \dots, u$; expand each point t times and add a set $H = \{a_1, a_2, \dots, a_h\}$. For $j = 1, 2, \dots, g$, let $p_{i,j}$ be the j -th partial parallel class which miss the group G_i ; for each $b \in p_{i,j}$, place on $b \times \{1, 2, \dots, t\}$ a copy $D_{i,j}^b$ of a (K_2, P_3, P_4) -URGDD(r_2, s_2, t_2) of type t^2 , with $(r_2, s_2, t_2) \in J_2$; place on $H \cup (G_i \times \{1, 2, \dots, t\})$ a copy D_i of a (K_2, P_3, P_4) -IURD($gt + h, h; [r_1, s_1, t_1], [r_3, s_3, t_3]$) with H as hole, $(r_1, s_1, t_1) \in J_1$ and $(r_3, s_3, t_3) \in J_3 = g * J_2$. Now combine all together the parallel classes of $D_{i,j}^b$, $b \in p_{i,j}$, along with the full classes of D_i so to obtain r_3 1-factors, s_3 classes of paths P_3 and t_3 classes of paths P_4 , $(r_3, s_3, t_3) \in J_3$, on $H \cup (\cup_{i=1}^u G_i \times \{1, 2, \dots, t\})$. Fill the hole H with a copy D of (K_2, P_3, P_4) -URD($h; r_1, s_1, t_1$) with $(r_1, s_1, t_1) \in J_1$ and combine the classes of D with the partial classes of D_i so to obtain r_1 1-factors, s_1 classes of paths P_3 and t_1 classes of paths P_4 on $H \cup (\cup_{i=1}^u G_i \times \{1, 2, \dots, t\})$. The result is a (K_2, P_3, P_4) -URD($v + h; r, s, t$) for each $(r, s, t) \in J_1 + u * J_3$. \square

4 Small cases

Lemma 4.1. *There exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 6^2 , for every $(r, s, t) \in \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$.*

Proof. The cases $(6, 0, 0), (3, 0, 2), (0, 0, 4)$ correspond to a (P_2, P_4) -URGDD(r_1, s_1) of type 6^2 , with $(r_1, s_1) \in \{(6, 0), (3, 2), (0, 4)\}$, which is known to exist [14]. For the case $(2, 3, 0)$ take the groups to be $\{1, 2, 3, 4, 5, 6\}, \{a, b, c, d, e, f\}$ and the classes listed below:

$\{\{1, e\}, \{2, f\}, \{3, c\}, \{4, d\}, \{5, b\}, \{6, a\}\}, \{\{1, d\}, \{2, b\}, \{3, a\}, \{4, e\}, \{5, f\}, \{6, c\}\},$
 $\{[a, 1, b], [f, 4, c], [3, d, 5], [2, e, 6]\}, \{[c, 2, d], [a, 5, e], [3, b, 4], [1, f, 6]\}, \{[e, 3, f], [b, 6, d],$
 $[2, a, 4], [1, c, 5]\}.$

□

Lemma 4.2. *There exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 4^3 , for every $(r, s, t) \in \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\}$.*

Proof. The case $(0, 6, 0)$ follows by [23]. For the remaining cases, take the groups to be $\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}$ and the classes listed below:

- $(4, 3, 0)$:
 $\{\{1, 6\}, \{2, 5\}, \{3, 10\}, \{4, 9\}, \{7, 12\}, \{8, 11\}\},$
 $\{\{1, 5\}, \{2, 6\}, \{3, 9\}, \{4, 10\}, \{7, 11\}, \{8, 12\}\},$
 $\{\{1, 9\}, \{2, 10\}, \{3, 7\}, \{4, 8\}, \{6, 12\}, \{5, 11\}\},$
 $\{\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 12\}, \{6, 11\}\},$
 $\{[8, 1, 11], [7, 2, 12], [4, 5, 9], [3, 6, 10]\}, \{[11, 3, 5], [12, 4, 6], [1, 7, 9], [2, 8, 10]\},$
 $\{[6, 9, 8], [5, 10, 7], [2, 11, 4], [1, 12, 3]\}.$
- $(1, 3, 2)$:
 $\{\{1, 6\}, \{2, 5\}, \{3, 10\}, \{4, 9\}, \{7, 12\}, \{8, 11\}\},$
 $\{[8, 1, 11], [7, 2, 12], [4, 5, 9], [3, 6, 10]\}, \{[11, 3, 5], [12, 4, 6], [1, 7, 9], [2, 8, 10]\},$
 $\{[6, 9, 8], [5, 10, 7], [2, 11, 4], [1, 12, 3]\},$
 $\{[9, 1, 10, 4], [2, 6, 11, 5], [12, 8, 3, 7]\}, \{[1, 5, 12, 6], [3, 9, 2, 10], [8, 4, 7, 11]\}.$

□

Lemma 4.3. $URD(12; P_2, P_3, P_4) \supseteq D(12).$

Proof. The case $((2, 0, 6))$ corresponds to a (P_2, P_4) -URD($12; 2, 6$) which is known to exist [14]. The case $(0, 6, 2)$ corresponds to a (P_3, P_4) -URD($12; 6, 2$) which is

known to exist [8]. For all the other cases take a (K_2, P_3, P_4) -URGDD(r, s, t) of type 6^2 with $(r, s, t) \in \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$, which exists by Lemma 4.1. Fill in each group of size 6 with a copy of a (P_2, P_3, P_4) -URD($6; r_1, s_1, t_1$) with $(r_1, s_1, t_1) \in \{(5, 0, 0), (1, 3, 0)\}$, which exists by Lemma 2.2. This gives a (K_2, P_3, P_4) -URD($12; r, s, t$) for every $(r, s, t) \in \{(5, 0, 0), (1, 3, 0)\} + \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$. \square

Lemma 4.4. *There exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^2 , for every $(r, s, t) \in \bar{D}(12^2)$.*

Proof. The case $(0, 9, 0)$ corresponds to a (P_3) -URGDD(9) of type 12^2 which is known to exist [22]. To obtain all remaining cases except $(1, 6, 2)$, start from a 2-RGDD of type 2^2 with the block set partitioned into two 1-factors, expand each point 6 times and for each edge e of a 1-factor place on $e \times \{1, 2, 3, 4, 5, 6\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 6^2 (from Lemma 4.1) so to obtain a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^2 , for every $(r, s, t) \in \bar{D}(12^2) \setminus \{(1, 6, 2), (0, 9, 0)\}$. For the case $(1, 6, 2)$ take the groups $\{0, 1, \dots, 11\}$ and $\{0', 1', \dots, 11'\}$ and the classes as listed below:

$\{\{0, 11'\}, \{1, 4'\}, \{2, 6'\}, \{3, 7'\}, \{4, 1'\}, \{5, 0'\}, \{6, 10'\}, \{7, 5'\}, \{8, 3'\}, \{9, 8'\}, \{10, 2'\}, \{11, 9'\}\},$
 $\{[0', 0, 1'], [2', 1, 3'], [6', 6, 7'], [8', 7, 9'], [2, 4', 3], [4, 5', 5], [8, 10', 9], [10, 11', 11]\},$
 $\{[1', 2, 5'], [2', 5, 4'], [7', 8, 11'], [8', 11, 10'], [1, 0', 3], [7, 6', 9], [0, 3', 4], [6, 9', 10]\},$
 $\{[4', 4, 0'], [5', 3, 3'], [10', 10, 6'], [11', 9, 9'], [1, 1', 5], [0, 2', 2], [7, 7', 11], [6, 8', 8]\},$
 $\{[0', 6, 1'], [2', 7, 3'], [6', 0, 7'], [8', 1, 9'], [2, 10', 3], [4, 11', 5], [8, 4', 9], [10, 5', 11]\},$
 $\{[1', 8, 5'], [2', 11, 4'], [7', 2, 11'], [8', 5, 10'], [1, 6', 3], [7, 0', 9], [0, 9', 4], [6, 3', 10]\},$
 $\{[4', 10, 0'], [5', 9, 3'], [10', 4, 6'], [11', 3, 9'], [1, 7', 5], [0, 8', 2], [7, 1', 11], [6, 2', 8]\},$
 $\{[4', 0, 5', 1], [6, 11', 7, 10'], [0', 2, 3', 5], [11, 6', 8, 9'], [1', 3, 2', 4], [9, 7', 10, 8']\},$
 $\{[7, 4', 6, 5'], [0, 10', 1, 11'], [8, 0', 11, 3'], [2, 9', 5, 6'], [3, 8', 4, 7'], [10, 1', 9, 2']\}.$ \square

Lemma 4.5. *There exists a (P_2, P_3, P_4) -URGDD($1, 6, 10$) of type 12^3 .*

Proof. Let $G_i = 3Z_{36} + i$, $i = 0, 1, 2$. We construct a (P_2, P_3, P_4) -URGDD($1, 6, 10$) of type 12^3 by listing its classes as follows:

$\{\{1 + 4i, 2 + 4i\}, \{3 + 4i, 8 + 4i\} : i = 0, 1, \dots, 8, i \in Z_{36}\}$
 $\{[3i, 2 + 3i, 4 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$
 $\{[1 + 3i, 3 + 3i, 11 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$
 $\{[3i, 4 + 3i, 8 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$
 $\{[6 + 3i, 2 + 3i, 10 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$

$\{[9 + 3i, 1 + 3i, 17 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$
 $\{[34 + 3i, 18 + 3i, 2 + 3i] : i = 0, 1, \dots, 11, i \in Z_{36}\},$
 $\{[2 + 4i, 3 + 4i, 4 + 4i, 5 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[4i, 5 + 4i, 10 + 4i, 15 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[4i, 7 + 4i, 14 + 4i, 1 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[11 + 4i, 1 + 4i, 8 + 4i, 18 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[12 + 4i, 2 + 4i, 9 + 4i, 23 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[14 + 4i, 3 + 4i, 13 + 4i, 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[11 + 4i, 4i, 14 + 4i, 25 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[12 + 4i, 1 + 4i, 18 + 4i, 31 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[2 + 4i, 19 + 4i, 32 + 4i, 13 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\},$
 $\{[17 + 4i, 3 + 4i, 20 + 4i, 6 + 4i] : i = 0, 1, \dots, 8, i \in Z_{36}\}.$

□

Lemma 4.6. *There exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^3 , for every $(r, s, t) \in \bar{D}(12^3)$.*

Proof. The case $(0, 9, 8)$ corresponds to a (P_2, P_3, P_4) -URGDD($0, 9, 8$) of type 12^3 which is known to exist [8], while the case $(1, 6, 10)$ is given by Lemma 4.5.

For the cases $(4, 15, 0)$, $(0, 18, 0)$, $(5, 12, 2)$, $(1, 15, 2)$, $(2, 12, 4)$, $(3, 9, 6)$, start from a 3-RGDD \mathcal{D} of type 3^3 with three parallel classes, expand each point 4 times and for each block b of a given parallel class of \mathcal{D} place on $b \times \{1, 2, 3, 4\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 4^3 , with $(r_1, s_1, t_1) \in \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\}$, which exists by Lemma 4.2. Since \mathcal{D} contains three parallel classes the result is a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^3 , for every $(r, s, t) \in 3 * \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\} \supseteq \{(4, 15, 0), (0, 18, 0), (5, 12, 2), (1, 15, 2), (2, 12, 4), (3, 9, 6)\}$.

To settle the remaining cases, start from a 2-RGDD of type 2^3 with the block set partitioned into four 1-factors, expand each point 6 times and for each edge e of a 1-factor place on $e \times \{1, 2, 3, 4, 5, 6\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 6^2 , with $(r_1, s_1, t_1) \in \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$, which exists by Lemma 4.1. The result is a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^3 , for every $(r, s, t) \in 4 * \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\} \supseteq \{\bar{D}(12^3) \setminus \{(4, 15, 0), (0, 18, 0), (5, 12, 2), (1, 15, 2), (2, 12, 4), (3, 9, 6), (0, 9, 8), (1, 6, 10)\}\}$. □

Lemma 4.7. $URD(36; P_2, P_3, P_4) \supseteq D(36)$.

Proof. Start from a 3-RGDD of type 1^3 and apply Theorem 3.4 with $t = 12, g = 1, u = 3$ (the input designs are a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 12^3 with

$(r_1, s_1, t_1) \in \bar{D}(12^3)$, which exists by Lemma 4.6, and a (P_2, P_3, P_4) -URD $(12; r_2, s_2, t_2)$ with $(r_2, s_2, t_2) \in D(12)$, which exists by Lemma 4.3). This implies

$$URD(v; P_2, P_3, P_4) \supseteq D(12) + \bar{D}(12^3).$$

Taking into account that

$$D(12) = D_0(12) \cup D_1(12) \cup D_2(12)$$

and

$$\bar{D}(12^3) = \bar{D}_0(12^3) \cup \bar{D}_1(12^3) \cup \dots \cup \bar{D}_6(12^3).$$

Since it is easy to see that, for each $j = 0, 1, \dots, 6$

$$\bar{D}_j(12^3) + D_2(12) = D_{j+2}(36)$$

and, for each $i = 0, 1$,

$$\bar{D}_0(12^3) + D_i(12) = D_i(36),$$

we obtain $D(12) + \bar{D}(12^3) = D(36)$ and this complete the proof. \square

Lemma 4.8. *There exists a (P_2, P_3, P_4) -IURD $(36, 12; [r_1, s_1, t_1], [r_2, s_2, t_2])$ for every $(r_1, s_1, t_1) \in D(12)$ and for every $(r_2, s_2, t_2) \in 2 * \bar{D}(12^2)$.*

Proof. Lemma 3.3 gives $2 * \bar{D}(12^2) = \bar{D}(12^3)$. Start from a (P_2, P_3, P_4) -URGDD (r_2, s_2, t_2) of type 12^3 with $(r_2, s_2, t_2) \in \bar{D}(12^3)$, which exists by Lemma 4.6, and fill in two groups of size 12 with a copy of a (P_2, P_3, P_4) -URD $(12; r_1, s_1, t_1)$ with $(r_1, s_1, t_1) \in D(12)$, which exists by Lemma 4.3. \square

In order to obtain our main result, we need to handle a further case, $v = 60$, which will be discussed in a separate section.

4.1 The case $v = 60$

Lemma 4.9. [14] $URD-(20; P_2, P_3, P_4) = D(20) = \{(19 - 3x, 0, 2x), x = 0, 1, \dots, 6\}$.

In what follows, we will denote by $C_{m(n)}$ the graph $G_{(n)}$ where G is a m -cycle (a m -cycle C_m with vertex set $\{a_1, a_2, \dots, a_m\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{m-1}, a_m\}, \{a_m, a_1\}\}$ will be denoted by (a_1, a_2, \dots, a_m)).

Lemma 4.10. *There exists a resolvable P_3 -decomposition of $C_{5(6)}$.*

Proof. On $Z_6 \times Z_5$ consider the set \mathcal{B} of copies of P_3 obtained by developing in Z_6 the following base blocks (partitioned into three sets, for convenience):

$$\mathcal{B}_0^{(1)}: [0_1, 0_0, 1_1], [2_2, 2_1, 3_2], [1_3, 1_2, 2_3], [2_4, 0_3, 3_4], [1_0, 1_4, 2_0];$$

$$\mathcal{B}_0^{(2)}: [2_1, 0_0, 3_1], [3_2, 1_1, 4_2], [4_3, 2_2, 5_3], [4_4, 0_3, 5_4], [4_0, 0_4, 5_0];$$

$$\mathcal{B}_0^{(3)}: [4_1, 0_0, 5_1], [4_2, 0_1, 5_2], [4_3, 0_2, 5_3], [0_4, 0_3, 1_4], [4_0, 2_4, 5_0].$$

For $j = 1, 2, 3$ and $i \in Z_6$, let $\mathcal{B}_i^{(j)} = \{b + i : b \in \mathcal{B}_0^{(j)}\}$. The blocks of \mathcal{B} can be partitioned into the nine parallel classes $\mathcal{B}_i^{(j)} \cup \mathcal{B}_{3i}^{(j)}$, for $j = 1, 2, 3$ and $i = 0, 1, 2$. \square

Lemma 4.11. *There exists a (P_2, C_5) -URGDD(s, t) of type 2^5 , for every $(s, t) \in \{(8, 0), (6, 1), (4, 2)\}$.*

Proof. The case $(8, 0)$ is trivial.

Take Z_{10} as vertex set and $G_i = 5Z_{10} + i$, $i = 0, 1, 2, 3, 4$ as groups; the classes listed below:

- $(6, 1)$:
 - $\{\{2i, 1 + 2i\} : i = 1, 2, 4, i \in Z_{10}\} \cup \{\{0, 6\}, \{1, 7\}\},$
 - $\{\{1 + 2i, 2 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{2i, 3 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{1 + 2i, 4 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{0, 4\}, \{8, 2\}, \{1, 5\}, \{9, 3\}, \{6, 7\}\},$
 - $\{\{4, 8\}, \{2, 6\}, \{5, 9\}, \{3, 7\}, \{0, 1\}\},$
 - $\{(i, 2 + i, 4 + i, 6 + 1, 8 + i) : i = 0, 1, i \in Z_{10}\}.$
- $(4, 2)$:
 - $\{\{2i, 1 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{1 + 2i, 2 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{2i, 3 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{\{1 + 2i, 4 + 2i\} : i = 0, 1, 2, 3, 4, i \in Z_{10}\},$
 - $\{(i, 2 + i, 4 + i, 6 + i, 8 + i) : i = 0, 1, i \in Z_{10}\}$
 - $\{(i, 4 + i, 8 + i, 2 + i, 6 + i) : i = 0, 1, i \in Z_{10}\}.$

\square

Define the following set of triples:

$$\begin{aligned} I_1 &= \{(48 - 3x, 0, 2x), x = 0, 1, \dots, 16\}, I_2 = \{(44 - 3x, 3, 2x), x = 0, 1, \dots, 14\}, \\ I_3 &= \{(36 - 3x, 9, 2x), x = 0, 1, \dots, 12\}, I_4 = \{(32 - 3x, 12, 2x), x = 0, 1, \dots, 10\}, \\ I_5 &= \{(24 - 3x, 18, 2x), x = 0, 1, \dots, 8\}, I_6 = \{(20 - 3x, 21, 2x), x = 0, 1, \dots, 6\}, \\ I_7 &= \{(0, 36, 0)\}, I_8 = \{(4, 33, 0), (1, 33, 2)\}. \end{aligned}$$

Lemma 4.12. *There exists a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^5 , for every $(r, s, t) \in \cup_{j=1}^8 I_j$.*

Proof. Start from a (P_2, C_5) -URGDD(s, t) of type 2^5 [17], with $(s, t) \in \{(8, 0), (6, 1), (4, 2)\}$, which exists by Lemma 4.11, expand each point 6 times and for every block b of a parallel class place on $b \times \{1, 2, 3, 4, 5, 6\}$ a copy of a (P_2, P_3, P_4) -URGDD(r, s, t) of type 6^2 with $(r_1, s_1, t_1) \in \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$, from Lemma 4.1, or, as the case may be, a copy of a resolvable P_3 -decomposition of $C_{5(6)}$ from Lemma 4.10. This gives a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^5 with $(r, s, t) \in A_1 \cup A_2 \cup A_3$, where $A_1 = 8 * \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\}$, $A_2 = 6 * \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\} + (0, 9, 0)$, $A_3 = 4 * \{(6, 0, 0), (3, 0, 2), (0, 0, 4), (2, 3, 0)\} + (0, 18, 0)$, and since $I_i \cup I_{i+1} \subseteq A_{\frac{i+1}{2}}$, $i = 1, 3, 5$, $(r, s, t) \in \cup_{j=1}^6 I_j$.

To obtain the remaining triples in $I_7 \cup I_8$, start from a 3-RGDD of type 3^5 , expand each point 4 times and for every block b of a given parallel class place on $b \times \{1, 2, 3, 4\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 4^3 , with $(r_1, s_1, t_1) \in \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\}$, which exists by Lemma 4.2. The result is a (P_2, P_3, P_4) -URGDD(r, s, t) of type 12^5 with $(r, s, t) \in 6 * \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\} \supseteq I_i$, for $i = 7, 8$, and this completes the proof. \square

Lemma 4.13. *There exists a (P_2, P_3, P_4) -URGDD($0, 30, 0$) of type 20^3 .*

Proof. Start from a 3-RGDD of type 5^3 , expand every point 4 times and for every block b of a given parallel class place on $b \times \{1, 2, 3, 4\}$ a copy of a (P_2, P_3, P_4) -URGDD(r_1, s_1, t_1) of type 4^3 , with $(r_1, s_1, t_1) \in \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\}$, which exists by Lemma 4.3. The result is a (P_2, P_3, P_4) -URGDD(r, s, t) of type 20^3 with $(r, s, t) \in 5 * \{(0, 6, 0), (4, 3, 0), (1, 3, 2)\} \supseteq \{(0, 30, 0)\}$. \square

Lemma 4.14. $URD(60; P_2, P_3, P_4) \supseteq D(60)$.

Proof. The case $(0, 33, 10)$ corresponds to a (P_2, P_3, P_4) -URD($60; 0, 33, 10$) which is known to exist [8]. For $D_{10}(60)$, start from a (P_2, P_3, P_4) -URGDD($0, 30, 0$) of type 20^3 , which exists by Lemma 4.13, and fill in each group with a copy of a (P_2, P_3, P_4) -URD($20; r_2, 0, t_2$) from Lemma 4.9 so to obtain the set of triples $I_9 + D(20) = D_{10}(60)$.

For all the other cases, start from a (P_2, P_3, P_4) -URGDD (r_1, s_1, t_1) of type 12^5 from Lemma 4.12 and fill in each group with a copy of a (P_2, P_3, P_4) -URD $(12; r_2, s_2, t_2)$ from Lemma 4.3 so to obtain the following triples:

$$\begin{aligned} I_1 + D_i(12) &= D_i(60), \quad i = 0, 1, 2, \\ I_2 + D_2(12) &= D_3(60), \quad I_3 + D_1(12) = D_4(60), \quad I_3 + D_2(12) = D_5(60), \\ I_5 + D_i(12) &= D_{i+6}(60), \quad i = 0, 1, 2, \\ I_6 + D_2(12) &= D_9(60), \quad I_8 + D_0(12) = D_{11} - \{(0, 33, 10)\}, \\ I_7 + D_i(12) &= D_{i+12}(60), \quad i = 0, 1, 2. \end{aligned}$$

□

5 The case $v \equiv 0 \pmod{24}$

Lemma 5.1. *For every $v \equiv 0 \pmod{24}$, $URD(v; P_2, P_3, P_4) \supseteq D(v)$.*

Proof. Start from a 2-RGDD of type $1^{\frac{v}{12}}$ [3] and apply Theorem 3.4 with $t = 12, g = 1, u = \frac{v}{12}$ (the input designs are a (P_2, P_3, P_4) -URGDD (r_1, s_1, t_1) of type 12^2 with $(r_1, s_1, t_1) \in \bar{D}(12^2)$, which exists by Lemma 4.4, and a (P_2, P_3, P_4) -URD $(12; r_2, s_2, t_2)$ with $(r_2, s_2, t_2) \in D(12)$, which exists by Lemma 4.3). This implies

$$URD(v; P_2, P_3, P_4) \supseteq D(12) + \frac{v-12}{12} * D(12^2)$$

. Since

$$D(12) = D_0(12) \cup D_1(12) \cup D_2(12)$$

and

$$\frac{v-12}{12} * \bar{D}(12^2) = \bar{D}((v-12)^2) = \bar{D}_0((v-12)^2) \cup \bar{D}_1((v-12)^2) \cup \dots \cup \bar{D}_{\frac{v-12}{4}}((v-12)^2),$$

and, as it is easy to see, for each $j = 0, 1, \dots, \frac{v-12}{4}$

$$\bar{D}_j((v-12)^2) + D_2(12) = D_{j+2}(v)$$

and, for each $i = 0, 1,$

$$\bar{D}_0((v-12)^2) + D_i(12) = D_i(v),$$

we obtain

$$D(12) + \frac{v-12}{12} * D(12^2) = \bigcup_{j=0}^{\frac{v-4}{4}} D_j(v) = D(v).$$

□

6 The case $v \equiv 12 \pmod{24}$

Lemma 6.1. *For every $v \equiv 12 \pmod{24}$, $D(v) \subseteq URD(v; P_2, P_3, P_4)$.*

Proof. The cases $v = 12, 36, 60$ are covered by Lemmas 4.3, 4.7 and 4.14. For $v > 60$ start from a 2-frame of type $6^{\frac{v-12}{24}}$ ([20]) and apply Theorem 3.5 with $g = 6, u = \frac{v-12}{24}, t = 12$ and $h = 12$ (the input designs are: a (P_2, P_3, P_4) -URD($12; r_1, s_1, t_1$) with $(r_1, s_1, t_1) \in D(12)$, which exists by Lemma 4.3; a (P_2, P_3, P_4) -URGDD(r_2, s_2, t_2) of type 12^2 with $(r_2, s_2, t_2) \in \bar{D}(12^2)$, which exists by Lemma 4.4; a (P_2, P_3, P_4) -IURD($36, 12; [r_1, s_1, t_1], [r_3, s_3, t_3]$) with $(r_1, s_1, t_1) \in D(12)$ and $(r_3, s_3, t_3) \in 2 * \bar{D}(12^2)$, which exists by Lemma 4.8). This implies

$$URD(v; P_2, P_3, P_4) \supseteq D(12) + \frac{v-12}{24} * (2 * \bar{D}(12^2)).$$

Since

$$D(12) = D_0(12) \cup D_1(12) \cup D_2(12)$$

and

$$\begin{aligned} \frac{v-12}{24} * (2 * \bar{D}(12^2)) &= \\ &= \bar{D}((v-12)^2) \cup \bar{D}_0((v-12)^2) \cup \bar{D}_1((v-12)^2) \cup \dots \cup \bar{D}_{\frac{v-12}{4}}((v-12)^2). \end{aligned}$$

and, as it is easy to see, for each $j = 0, 1, \dots, \frac{v-12}{4}$

$$\bar{D}_j((v-12)^2) + D_2(12) = D_{j+2}(v)$$

and, for each $i = 0, 1$,

$$\bar{D}_0((v-12)^2) + D_i(12) = D_i(v),$$

we obtain

$$D(12) + \frac{v-12}{24} * (2 * \bar{D}(12^2)) = \bigcup_{j=0}^{\frac{v-4}{4}} D_j(v) = D(v).$$

□

7 Conclusion

We are now in a position to prove the following main result.

Theorem 7.1. *For every $v \equiv 0 \pmod{12}$, $URD(v; P_2, P_3, P_4) = D(v)$.*

Proof. Necessity follows by Lemmas 2.4. Sufficiency follows by Lemmas 6.1 and 5.1. \square

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